On the relation between turnpike properties and dissipativity for continuous time linear quadratic optimal control problems

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Abstract: The paper is devoted to analyze the connection between turnpike phenomena and strict dissipativity properties for continuous-time finite dimensional linear quadratic optimal control problems. We characterize strict dissipativity properties of the dynamics in terms of the system matrices related to the linear quadratic problem. These characterizations then lead to new necessary conditions for the turnpike properties under consideration, and thus eventually to necessary and sufficient conditions in terms of spectral criteria and matrix inequalities. One of the key novelty of these results is the possibility to encompass the presence of state and input constraints.

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1 Introduction

Turnpike phenomena refer to the property of optimal trajectories over finite but long time horizon to approach a steady state of the system and stay close to it during most of the optimally controlled evolution. Such behavior has been first observed and investigated in the context of optimal growth strategies towards economic equilibria by von Neumann [24] and in the book by Dorfman, Samuelson and Solow [6], where also the name "turnpike property" was coined. Following those results, turnpike phenomena have received a lot of attention in mathematical economy, see, e.g., [17]. Moreover, turnpike has been observed in different contexts, for example in mathematical biology [16] and chemical processes [20], and the phenomenon has been extensively studied from a mathematical point of view, e.g., in [3, 28, 29]. A key feature of the turnpike phenomenon consists in the structural insight one may deduce about the optimal solutions, for instance as a method for synthesizing long term optimal trajectories [1, 13, 19, 23] or for analyzing stability of model predictive control schemes [9], [12, Chapter 8].

This paper is devoted to analyze this property for finite-dimensional continuous-time optimal control problems with linear dynamics and a quadratic cost function, subject to input and state constraints. Here, the optimal control problem is not necessarily strictly convex, i.e., the quadratic term in the cost is only positive semidefinite.

In continuous-time, several sufficient conditions have been developed to ensure the turnpike property, even in an exponential form, based on different methods, for example exploiting Riccati-type characterizations combined with a Hamilton-Jacobi approach [1], or using the controllability of the problem and associated Riccati equations [19, 23], or relying on geometric considerations on the transversality of the stable and unstable manifolds [21]. All these references crucially rely on the hyperbolicity of the optimality system, which is at the base of the turnpike results. An alternative notion that allows to characterize the long time behavior of optimally controlled systems is the dissipativity of the system with respect to a given storage function, as introduced by Willems [25, 26, 27]. For discrete time systems it is known that this property is closely linked to the turnpike property, see [11].

Our focus in this paper is to link turnpike and strict dissipativity properties for continuous time problems, in terms of conditions on the matrices involved in the optimal control problem. This work represents the continuous-time counterpart of our previous work [10] in the discrete-time setting. The main differences with respect to that paper are the following. First, the different time evolution structure in this paper produces a different Lyapunov equation — see (4.1) — for characterizing the dissipativity of the system. Second, when passing to continuous time, several of the proofs in [10] need to be substantially reworked; particularly this concerns the proofs of Lemma 5.4, Theorem 7.1 and Lemma A. Finally, compared to [10], we cannot in general conclude *exponential* turnpike in all the settings discussed in this paper. This is because the results from [5] used in [10] are only available in discrete time. However, at least in the unconstrained situation we can replace those results by other recent ones from [21].

Similar relationships between dissipativity properties, optimal operation at steady state, and turnpike properties have been investigated in [7], where the authors show, on one hand, that dissipativity of the system implies both optimal operation at steady state and a turnpike property of optimal solutions and, on the other hand, they derive converse turnpike results, showing that under mild assumptions a turnpike of the optimal control problem implies dissipativity of the system. Compared to [7], in this paper, thanks to the specific linear-quadratic structure of the control problem, we are able not only to characterize turnpike and strict dissipativity of the problem in terms of system theoretical properties of the system such as detectability and stabilizability, but moreover to link them to the solvability of suitable matrix inequalities. This characterization then leads to precise spectral conditions on the matrices of the system to ensure such properties, and allows to point out the role played by the presence of state and input constraints in connection with the turnpike. In this regard, as already noted in [10], there are (at least) three conceptually different situations how the turnpike property interacts with state constraints: In the first case the turnpike phenomenon occurs both with and without constraints, provided the turnpike equilibrium lies inside the set of admissible states and controls. In the second situation the turnpike phenomenon only occurs if state constraints are present, but the location of the turnpike equilibrium is independent of the particular form of the constraints. Finally, in the third situation the position of the turnpike equilibrium depends on the constraint sets. In this paper we investigate the first and the second situation, illustrating them by several examples in Section 8. The third situation, briefly illustrated by Example 8.9, will be addressed in future research.

The remainder of the paper is organized as follows. In Section 2 we describe the optimal control problem, and we define the turnpike properties and the dissipativity properties considered in this paper. Section 3 shows that strict (pre)-dissipativity implies turnpike properties. Section 4 introduces a matrix inequality characterization of strict (pre)-dissipativity. Sections 5 and 6 reformulate this inequality in terms of the system matrices. The results in Section 7 show that turnpike properties imply strict (pre)-dissipativity. Finally, the main results and illustrative examples are collected in Section 8. A technical auxiliary result is stated and proved in the Appendix.

2 Setting and preliminaries

We consider the linear quadratic optimal control problems

$$\underset{u\in L^2(0,T;\mathbb{R}^m)}{\text{minimize}}\ J_T(x_0,u) \tag{2.1}$$

where

$$J_T(x_0,u) := \int_0^T \big(x(t)^TQx(t) + u(t)^TRu(t) + s^Tx(t) + v^Tu(t) + c\big)\,\mathrm{d}t\,,$$

$$\dot{x}(t) = Ax(t) + Bu(t)\,,\quad x(0) = x_0\,, \tag{2.2}$$

$T > 0$, $n, m \in \mathbb{N}$, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $Q \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times m}$, with Q and R symmetric, $Q \geq 0$ and $R > 0$, and $s \in \mathbb{R}^n$, $v \in \mathbb{R}^m$, $c \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$.

In what follows, we consider the optimal control problem (2.1) under input constraints $\mathbb{U} \subset \mathbb{R}^m$ and state constraints $\mathbb{X} \subseteq \mathbb{R}^n$, with both $\mathbb{X}$ and $\mathbb{U}$ being closed sets. To this end, for each $x_0 \in \mathbb{X}$ we define the space of admissible controls

$$\mathbb{U}^T(x_0) := \{u \in L^2(0,T;\mathbb{R}^m)\,|\,x_u(t,x_0) \in \mathbb{X} \text{ and } u(t) \in \mathbb{U} \text{ for a.e. } t \in (0,T)\}.$$

Here and in the following we denote by $x_u(\cdot,x_0)$ the solution of (2.2) with control u and initial value x_0.

We abbreviate the dynamics $f(x,u) := Ax + Bu$ and the stage cost as

$$\ell(x,u) := x^TQx + u^TRu + s^Tx + v^Tu + c\,.$$

We define the optimal value function

$$V_T(x_0) := \inf_{u\in\mathbb{U}^T(x_0)} J_T(x_0,u)$$

using the convention $V^T(x_0) = \infty$ if $\mathbb{U}^T(x_0) = \emptyset$. We call a control sequence $u^\star(\cdot) \in \mathbb{U}^T(x_0)$ and the corresponding trajectory $x^\star(\cdot,x_0)$ optimal if $J_T(x_0,u^\star) = V_T(x_0)$ holds. Moreover, we say that $(x^e,u^e) \in \mathbb{X}\times\mathbb{U}$ is an equilibrium for the dynamics f if it satisfies $f(x^e,u^e) = 0$.

The following definition specifies the two versions of the turnpike we study in this paper. Therein, μ denotes the Lebesgue-measure on $\mathbb{R}$.

Definition 2.1: (i) We say that the optimal control problem (2.1)-(2.2) has the *turnpike property* at an equilibrium $(x^e, u^e) \in \mathbb{X} \times \mathbb{U}$ on a set $\mathbb{X}_{tp} \subset \mathbb{X}$, if for each compact set $K \subset \mathbb{X}_{tp}$ and for each $\varepsilon > 0$ there exists a constant $C_{K,\varepsilon} > 0$ such that for all $x \in K$ and all $\delta > 0$ and all $T > 0$ the optimal trajectories $x^\star(\cdot, x)$ of (2.1) satisfy

$$\mu\big\{t \in (0,T) \,\big|\, \|x^\star(t,x) - x^e\| > \varepsilon\big\} \le C_{K,\varepsilon}\,.$$

(ii) We say that the optimal control problem (2.1)-(2.2) has the *near equilibrium turnpike property* at an equilibrium $(x^e, u^e) \in \mathbb{X} \times \mathbb{U}$, if for each $\rho > 0$, $\varepsilon > 0$ and $\delta > 0$ there exists a constant $C_{\rho,\varepsilon,\delta} > 0$ such that for all $x \in \mathbb{X}$ with $\|x - x^e\| \le \rho$, all $T > 0$, and all trajectories $x_u(\cdot, x)$ satisfying

$$J_T(x,u) \le T\ell(x^e, u^e) + \delta \tag{2.3}$$

for some $u \in \mathbb{U}$, the inequality

$$\mu\big\{t \in (0,T) \,\big|\, \|x_u(t,x) - x^e\| > \varepsilon\big\} \le C_{\rho,\varepsilon,\delta}\,.$$

holds.

In words, these properties state that the optimal/near equilibrium trajectories stay in an ε-neighbourhood of x^e for all but a set of "exceptional" time instants that is limited to transition intervals and whose Lebesgue-measure is bounded independently of the optimization horizon T.

Remark 2.2: (i) If the equilibrium (x^e, u^e) lies in the interior of $\mathbb{X} \times \mathbb{U}$, then Definition 2.1(i) implies that (A, B) is stabilizable, because otherwise there would be a subspace of initial conditions $\bar{x}_0$ with $\|x_u(t, \bar{x}_0)\| \ge \varepsilon(\bar{x}_0) > 0$ for all $t \ge 0$ and with $\varepsilon(\bar{x}_0)$ independent of the control function u. By choosing $\bar{x}_0$ sufficiently close to 0, this would imply the existence of $x_0 := \bar{x}_0 + x^e \in \mathbb{X}$ (sufficiently close to x^e and hence contained in $\mathbb{X}$) such that $\|x_u(t, x_0) - x^e\| = \|x_{u-u^e}(t, \bar{x}_0)\| \ge \varepsilon(\bar{x}_0) > 0$ for all control sequences $u \in \mathbb{U}$ and all $t \ge 0$. This contradicts the turnpike property.

(ii) In contrast to (i), Definition 2.1(ii) does not imply stabilizability, because there may not be nontrivial trajectories other than $x(t) \equiv x^e$, $u(t) \equiv u^e$ satisfying the assumed inequality (2.3) for J_T. A simple example for such a system is $\dot{x}(t) = x(t)$ with $\ell(x, u) = u^2$ and $x^e = u^e = 0$.

(iii) If (A, B) is stabilizable, then Definition 2.1(ii) implies Definition 2.1(i) provided (x^e, u^e) lies in the interior of $\mathbb{X} \times \mathbb{U}$. This is because stabilizability implies the existence of a stabilizing feedback law F such that the control $u(t) = F(x(t) - x^e) + u^e$ yields $x_u(t, x_0) \to x^e$ exponentially fast and $x_u(t, x_0) \in \mathbb{X}$, $u(t) \in \mathbb{U}$ for all $t \ge 0$ if x_0 lies in a sufficiently small neighbourhood $\mathcal{N}$ of x^e. This implies the existence of $C > 0$ with $V_T(x) \le T\ell(x^e, u^e) + C$ for all $x \in \mathcal{N}$. Hence, choosing $\mathbb{X}_{tp} = \mathcal{N}$, all optimal trajectories starting in $\mathbb{X}_{tp}$ satisfy the conditions of Definition 2.1(ii) and thus the turnpike property holds.

(iv) The statement from (iii) remains true in case (x^e, u^e) lies on the boundary $\partial(\mathbb{X} \times \mathbb{U})$ of $\mathbb{X} \times \mathbb{U}$ if for each $x \in \mathbb{X}$ sufficiently close to x^e there exists an admissible control u_x with $x_{u_x}(t, x) \to x_e$ and $u_x(t) \to u^e$ as $t \to \infty$, both exponentially fast. However, in contrast to (iii), for x^e or u^e not lying in the interior of the respective constraint set, the existence of such a u_x cannot in general be concluded from stabilizability of (A, B).

(v) We conjecture that most of our results in this paper remain true under additional terminal constraints on $x(T)$. Under particular terminal constraints, such as $x(T) = x^e$, it may even be possible to strengthen some of the results. However, in order not to overload the presentation we will not address this topic in this paper.

In other words, part (iii) of the remark says that the near equilibrium turnpike property plus stabilizability implies the turnpike property.

So far we have not specified how fast the number $C_{K,\varepsilon}$ in the turnpike property grows if $\varepsilon \to 0$, or, equivalently, how fast $\varepsilon > 0$ shrinks when we allow $C_{K,\varepsilon}$ to grow (always for fixed compact set $K \subset \mathbb{X}_{tp}$). The following definition describes an exponential form of the turnpike property.

Definition 2.3: We say that the turnpike property from Definition 2.1(i) is *exponential*, if there is $\theta \in (0,1)$ such that for each compact set $K \subset \mathbb{X}_{tp}$ there is a constant $m_K > 0$ such that $C_{K,\varepsilon}$ can be chosen as

$$C_{K,\varepsilon} \le m_K + \log_\theta \varepsilon\,.$$

We note that this inequality is equivalent to $\varepsilon \le M_K e^{-\lambda C_{K,\varepsilon}}$ with $M_K = \theta^{-m_K}$ and $\lambda = -\log\theta > 0$. This shows that ε shrinks exponentially fast when the bound $C_{K,\varepsilon}$ on the measure of points far from the turnpike grows.

The objective of this paper is to find easily checkable necessary and sufficient conditions on the data of the optimal control problem (2.1)-(2.2) (i.e., on A, B, Q, R, s, v and c) under which we can guarantee that turnpike properties hold. The next definitions provide the key concepts we use for this goal. For the definitions we recall that

$$\mathcal{K} := \{\alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \ : \ \alpha \text{ continuous, strictly increasing with } \alpha(0) = 0\}\,.$$

Definition 2.4: (i) We call the LQ problem *strictly pre-dissipative* at an equilibrium (x^e, u^e) on a set $X \subseteq \mathbb{R}^n$, $U \subseteq \mathbb{R}^m$ if there exist a storage function $\lambda : X \to \mathbb{R}$ which is bounded on bounded subsets of X and of class C^1, a function $\alpha \in \mathcal{K}$ and an equilibrium $(x^e, u^e) \in \mathbb{R}^n \times \mathbb{R}^m$ which satisfy the inequality

$$D\lambda(x) f(x,u) \le \ell(x,u) - \ell(x^e,u^e) - \alpha(\|x - x^e\|) \tag{2.4}$$

for all $x \in X$ and $u \in U$.

(ii) The system is called *strictly dissipative* at an equilibrium (x^e, u^e) on a set $X \subseteq \mathbb{R}^n$, $U \subseteq \mathbb{R}^m$ if it is strictly pre-dissipative in the sense of (i) and λ is bounded from below on X.

Remark 2.5: For a given $T > 0$, integrating (2.4) over the time interval $[0,T]$, the pre-dissipativity property at an equilibrium (x^e, u^e) on a set $X \subseteq \mathbb{R}^n$, $U \subseteq \mathbb{R}^m$ is recast as

$$\lambda(x_u(T,x_0)) \le \lambda(x_0) + \int_0^T \left[\ell(x_u(\tau,x_0),u(\tau)) - \ell(x^e,u^e) - \alpha(\|x_u(\tau,x_0) - x^e\|)\right] \mathrm{d}\tau,$$

for all $x_0 \in X$ and all $u \in \mathbb{U}^T(x_0)$.

We note that strict pre-dissipativity on a set X implies that the problem is strictly dissipative on each bounded subset $\widetilde{X}$ of X. Moreover, it is straightforward to see that strict pre-dissipativity holds if and only if the following *modified cost function*

$$\tilde{\ell}(x,u) := \ell(x,u) - \ell(x^e,u^e) - D\lambda(x)f(x,u) \tag{2.5}$$

satisfies

$$\tilde{\ell}(x,u) \geq \alpha(\|x - x^e\|) \tag{2.6}$$

for all $x \in X$ and $u \in U$.

Remark 2.6: (i) Strict dissipativity as defined in Definition 2.4(ii) is a strengthened version of the systems theoretic notion of dissipativity introduced by Willems in [25] and further studied in [26, 27]. Strict dissipativity, which differs from mere dissipativity by including the $\mathcal{K}$-function α in the inequality, is briefly mentioned by Willems under the name strong dissipativity, but it became popular only quite recently in the context of economic model predictive control, see [8] and the references therein. To the best of our knowledge, strict pre-dissipativity was introduced in the paper [10] in the context of discrete-time linear-quadratic problems and is here defined for the first time for continuous-time systems. It is related to the (non strict) concept of cyclodissipativity which is discussed, e.g., in [18, Chapter 3].

(ii) Under additional conditions, the storage function λ can be used as a Lyapunov function for the equilibrium x^e, thus ensuring (asymptotic) stability or controllability of x^e. While there is a certain similarity between asymptotic stability and turnpike properties, we emphasize that in this paper we will neither require λ to be a Lyapunov function nor will we impose additional conditions on λ besides those from Definition 2.4.

3 Strict (pre-)dissipativity implies turnpike

In this section we show that strict dissipativity implies turnpike properties. The first result gives conditions under which the near equilibrium turnpike property holds.

Theorem 3.1: Consider the LQ-problem (2.1), (2.2) with state and control constraint sets $\mathbb{X} \subseteq \mathbb{R}^n$ and $\mathbb{U} \subseteq \mathbb{R}^m$. Assume that

(i) the problem is strictly dissipative at an equilibrium (x^e, u^e) *or*

(ii) the problem is strictly pre-dissipative at an equilibrium (x^e, u^e) and $\mathbb{X}$ is bounded.

Then the near equilibrium turnpike property holds at (x^e, u^e).

Proof. (ii) follows from (i) since strict pre-dissipativity with bounded $\mathbb{X}$ implies strict dissipativity. We are thus left to prove that (i) implies the near equilibrium turnpike. For any $x_0 \in \mathbb{X}$, $T > 0$ and $u \in \mathbb{U}^T(x_0)$, for $\tilde{\ell}$ from (2.5) we set

$$\tilde{J}_T(x_0,u) = \int_0^T \tilde{\ell}(x_u(\tau,x_0), u(\tau))\,\mathrm{d}\tau$$

and $C := \inf_{x \in X} \lambda(x) > -\infty$. For any $\delta > 0$ and any trajectories $x(\cdot) := x_u(\cdot, x_0)$ satisfying

$$J_T(x_0, u) \le T\ell(x^e, u^e) + \delta$$

we deduce that

$$\begin{aligned} \tilde{J}_T(x_0, u) &= J_T(x_0, u) - T\ell(x^e, u^e) - \lambda(x(T)) + \lambda(x_0) \\ &\qquad\qquad \le \delta + \lambda(x_0) - \lambda(x(T)) \le \delta + \lambda(x_0) - C\,. \end{aligned} \tag{3.1}$$

Now, for any $\varepsilon > 0$, we set

$$S_\varepsilon := \{\tau \in (0, T) : \|x(\tau) - x^e\| > \varepsilon\}\,.$$

In order to ensure the near equilibrium turnpike property at (x^e, u^e) we shall provide a positive constant $C_{x_0,\varepsilon,\delta}$ (independent of T) such that $\mu(S_\varepsilon) \le C_{x_0,\varepsilon,\delta}$. To this aim, we claim that $\mu(S_\varepsilon) \le \frac{\delta + \lambda(x_0) - C}{\alpha(\varepsilon)}$. Indeed, if it were not true, we would have that

$$\begin{aligned} \tilde{J}_T(x_0, u) &\ge \int_{S_\varepsilon} \alpha(\|x(\tau) - x^e\|)\,\mathrm{d}\tau + \int_{[0,T]\setminus S_\varepsilon} \alpha(\|x(\tau) - x^e\|)\,\mathrm{d}\tau \\ &\qquad \ge \int_{S_\varepsilon} \alpha(\|x(\tau) - x^e\|)\,\mathrm{d}\tau > \mu(S_\varepsilon)\,\alpha(\varepsilon) \ge \delta + \lambda(x_0) - C\,, \end{aligned}$$

since over S_ε we have that $\|x(\tau) - x^e\| > \varepsilon$ and α is strictly increasing. The last relation contradicts (3.1), thus the proof is complete. ∎

The following result extends the previous theorem to the turnpike property at an equilibrium (x^e, u^e), where we need the additional assumptions that (A, B) is stabilizable and (x^e, u^e) lies in the interior of $\mathbb{X} \times \mathbb{U}$.

Corollary 3.2: Consider the LQ-problem (2.1), (2.2) with state and control constraint sets $\mathbb{X} \subseteq \mathbb{R}^n$ and $\mathbb{U} \subseteq \mathbb{R}^m$. Assume that (A, B) is stabilizable and

(i) the problem is strictly dissipative at an equilibrium $(x^e, u^e) \in \operatorname{int}(\mathbb{X} \times \mathbb{U})$ *or*

(ii) the problem is strictly pre-dissipative at an equilibrium $(x^e, u^e) \in \operatorname{int}(\mathbb{X} \times \mathbb{U})$ and $\mathbb{X}$ is bounded.

Then the turnpike property holds at (x^e, u^e).

Proof. It follows immediately from Theorem 3.1 and Remark 2.2(iii). ∎

We note that in the case of $\mathbb{X}$ unbounded, strict pre-dissipativity in general does not imply the turnpike property, as shown by Example 8.6 below.

For some of the results in this section converse statements were obtained in [11] in discrete time and in [7] in continuous time, even for general nonlinear-nonquadratic optimal control problems. In this paper we will address converse results in Section 7. In the present continuous-time linear-quadratic setting we will be able to present stronger characterizations than those in [7, 11].

4 A matrix condition for strict (pre)-dissipativity

In this section we show that strict (pre-)dissipativity can be equivalently characterized in terms of matrix inequalities.

Lemma 4.1: Given $P \in \mathbb{R}^{n\times n}$, there exists $q \in \mathbb{R}^n$ such that the LQ problem is strictly pre-dissipative with storage function $\lambda(x) = x^T Px + q^T x$ if and only if the matrix inequality

$$Q - A^T P - PA > 0 \tag{4.1}$$

is satisfied. In particular, if the problem is strictly pre-dissipative for certain s, v and c, then the problem is strictly pre-dissipative for all s, v and c. Moreover, if P is positive definite then the problem is strictly dissipative.

Proof. The proof is based on the fact that strict pre-dissipativity holds if and only if the inequality (2.6), i.e., $\tilde{\ell}(x,u) \geq \alpha(\|x - x^e\|)$ holds for all $x \in X$ and $u \in U$ and the modified cost function $\tilde{\ell}$ from (2.5).

First assume that the system is strictly pre-dissipative with λ from the assumption. Then a straightforward computation yields that $\tilde{\ell}$ is of the form

$$\tilde{\ell}(x,u) = x^T(Q - A^T P - PA)x + R(x,u), \tag{4.2}$$

where $R(x,u)$ contains only terms that are linear or constant in x. Since $f(x^e,u^e) = 0$ we deduce that $\tilde{\ell}(x^e,u^e) = 0$, and inequality (2.6) implies that $x \mapsto \tilde{\ell}(x,u^e)$ has a strict local minimum in $x = x^e$. For a function of the form (4.2) this is only possible if the quadratic part is strictly convex, i.e, if $Q - A^T P - PA$ is positive definite.

Conversely, assume $Q - A^T P - PA > 0$. For a given $\gamma \in (0,1]$, set $P_\gamma := \gamma P$ and

$$Q_\gamma := Q - A^T P_\gamma - P_\gamma A\,,$$

which is positive definite since $Q_\gamma = (1-\gamma)Q + \gamma(Q - A^T P - PA) > 0$. Consider the modified stage cost

$$\ell_\gamma(x,u) := \ell(x,u) - x^T P_\gamma f(x,u) - f(x,u)^T P_\gamma x\,.$$

We claim that ℓ_γ is strictly convex in (x,u), for a suitable value of γ. Indeed,

$$\ell_\gamma(x,u) = x^T Q_\gamma x + u^T Ru - x^T P_\gamma Bu - u^T B^T P_\gamma x + R(x,u)\,,$$

where $R(x,u)$ contains lower order terms in (x,u). Setting $C := -PB - B^T P$, convexity of ℓ_γ is equivalent to positive definiteness of the matrix

$$H := \begin{pmatrix} 2Q_\gamma & \gamma C \\ \gamma C & 2R \end{pmatrix}.$$

Since R is positive definite, H is positive definite if and only if the Schur complement of $2R$ in H, that is, $S_\gamma := 2Q_\gamma - \gamma^2 C(2R)^{-1}C$, is positive definite. Since Q_γ is positive definite, the Schur complement S_γ is positive definite for sufficiently small γ. We thus conclude that, for a sufficiently small $\bar\gamma \in (0,1]$, the modified stage cost $\ell_{\bar\gamma}$ is strictly convex in (x,u), and the optimal equilibrium problem

$$\min_{x\in X, u\in U} \ell_{\bar\gamma}(x,u), \quad \text{s.t.} \quad x - Ax - Bu = 0,$$

admits a unique global solution (x^e, u^e). We thus deduce (see, e.g., [2, Section 5.9.1]) the existence of a vector $q \in \mathbb{R}^n$ such that the LQ-problem with stage cost $\ell_{\bar\gamma}$ is strictly pre-dissipative at (x^e,u^e) with storage function $\hat\lambda(x) = q^T x$. This implies that the LQ-problem with the original stage cost ℓ is strictly pre-dissipative with storage function $\lambda(x) = x^T P_{\bar\gamma}\, x + \hat\lambda(x) = x^T P_{\bar\gamma}\, x + q^T x$, indeed

$$\begin{aligned}\ell(x,u) - \ell(x^e,u^e) = \ell_{\bar\gamma}(x,u) - \ell_{\bar\gamma}(x^e,u^e) + x^T P_{\bar\gamma} f(x,u) + f(x,u)^T P_{\bar\gamma} x \\ \geq D\lambda(x) f(x,u) + \alpha(\|x - x^e\|),\end{aligned}$$

which proves the claim.

The assertion on s, v and c follows immediately because the matrix condition is independent of on s, v and c and also of x^e and u^e, which implicitly depend on on s, v and c. Finally, positive definiteness of P implies that the storage function $\lambda(x) = x^T Px + q^T x$ is bounded from below on the whole $\mathbb{R}^n$, hence the problem is strictly dissipative. □

Remark 4.2: Equation (4.1) is, in fact, a Lyapunov equation, see e.g. [14, eq. (80a) and (80b)]. The main difference here is that we allow for indefinite solutions P of the equation, while in the theory of Lyapunov equations positive definite solutions P are sought in order to ensure that $V(x) = x^T Px$ is a Lyapunov function. This difference is consistent with Remark 2.6(ii), i.e., with the fact that we do not require the storage function to be a Lyapunov function.

5 Observable and non-observable systems

In this section we derive necessary and sufficient conditions on the matrices A and C under which the matrix inequality (4.1) holds.

Definition 5.1: Consider a matrix pair (A,C) with $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{l\times n}$.
(i) We call $x_0 \in \mathbb{R}^n \setminus \{0\}$ *unobservable*, if the solutions of $\dot x(t) = Ax(t)$ with $x(0) = x_0$ satisfy $Cx(t) = 0$ for all $t \geq 0$. Otherwise we call x_0 *observable.*

(ii) We say that the matrix pair (A,C) is *observable*, if every $x_0 \in \mathbb{R}^n \setminus \{0\}$ is observable.

(iii) Let $x = w + iv \in \mathbb{C}^n \setminus \{0\}$ be an eigenvector of A corresponding to the eigenvalue μ. We say that x is an *unobservable eigenvector* if either w or v is unobservable. In this case, we call μ an *unobservable eigenvalue.*

(iv) We call (A,C) *detectable* if all unobservable eigenvalues μ satisfy $Re(\mu) < 0$.

One can show (for details see, e.g., [22, Chapter 6]) that x_0 is unobservable if and only if it lies in the kernel of the observability matrix $\mathbf{O}(A,C) := (C^T, (CA)^T, \ldots, (CA^{n-1})^T)^T$. This implies that (A,C) is observable if and only if the observability matrix has full rank. Another condition equivalent to observability is the Hautus criterion, which demands that the matrix

$$\begin{pmatrix} A - \mu I \\ C \end{pmatrix}$$

has full rank for all eigenvalues μ of A.

Remark 5.2: If $x = w+iv \in \mathbb{C}^n \setminus \{0\}$ is an eigenvector of A corresponding to an eigenvalue $\mu = a + ib$ with $b \neq 0$, then both w and v belong to $\mathbb{R}^n \setminus \{0\}$. Moreover, since C is real, if $x \in \mathbb{C}^n \setminus \{0\}$ is an unobservable eigenvector of A with eigenvalue μ, then its complex conjugate $\bar{x}_0 = w - iv$ is also an unobservable eigenvector of A corresponding to the eigenvalue $\bar{\mu}$. Finally, from relation (A.4) in the Appendix we deduce that w is observable if and only if v is observable. Therefore, if μ is an unobservable eigenvalue then both w and v are unobservable.

Remark 5.3: Let x_0 be either an unobservable real eigenvector or of the form $x_0 = w$ for an unobservable complex eigenvector $w + iv$, and let $Q = C^T C$. For any $\gamma \in \mathbb{R}$ and $u \in \mathbb{U}^\infty := \{(u(t))_{t\geq 0} : u(t) \in \mathbb{U}\ \forall t \geq 0\}$, the solution $x_u(t, \gamma x_0)$ is of the form

$$x_u(t, \gamma x_0) = \gamma e^{tA} x_0 + x_u(t, 0)\,, \qquad \forall t \geq 0\,. \tag{5.1}$$

Since $Ce^{tA}x_0 = 0$ for all $t \geq 0$ this implies

$$\begin{aligned}
&\ell(x_u(t,\gamma x_0), u)\\
=\ & x_u(t,\gamma x_0)^T Q x_u(t,\gamma x_0) + u(t)^T R u(t) + s^T x_u(t,\gamma x_0) + v^T u(t) + c\\
=\ & x_u(t,0)^T Q x_u(t,0) + u(t)^T R u(t) + s^T \gamma e^{tA} x_0 + s^T x_u(t,0) + v^T u(t) + c\\
=\ & \big[\underbrace{x_u(t,0)^T Q x_u(t,0) + u(t)^T R u(t) + s^T x_u(t,0) + v^T u(t)}_{=:\ell_1(t,u(t))}\big] + \big[\underbrace{s^T \gamma e^{tA} x_0 + c}_{=:\ell_2(t,\gamma x_0)}\big].
\end{aligned} \tag{5.2}$$

From the last expression one sees that the stage cost decomposes into a first part ℓ_1 which is independent of x_0 and γ and a second part ℓ_2 which is independent of u. Hence, the same holds for the optimization objective which can thus be written as

$$J_T(\gamma x_0, u) = \int_0^T \ell(x_u(t,\gamma x_0), u(t))\,\mathrm{d}t = \int_0^T \ell_1(t, u(t))\,\mathrm{d}t + \int_0^T \ell_2(t, \gamma x_0)\,\mathrm{d}t\,.$$

This implies that the optimal control u^* is independent of γ, except in the case when the state constraints require a change in the control action when γ changes.

The following lemmas establish relations between observability and spectral properties of A, respectively, and the solvability of (4.1). The proof of the first lemma uses an adaptation of an argument from [4].

Lemma 5.4: Consider the LQ-problem (2.1), (2.2) with $Q = C^T C$ and (A, C) detectable. Then there exists a symmetric and positive definite matrix P such that (4.1) holds.

Proof. We follow the ideas of [4, Lemma 1.7.3]. By duality, the detectability of (A, C) is equivalent to the stabilizability of the pair (A^T, C^T). Thus, there exists a matrix $F \in \mathbb{R}^{n\times n}$ such that $A^T + C^T F$ is asymptotically stable, i.e, there exists a symmetric and positive definite matrix X such that

$$(A^T + C^T F)X + X(A^T + C^T F)^T < 0\,.$$

In particular, for nonzero $x \in Ker(C)$, this implies that $x^T A^T X x + x^T X A x < 0$. Then $Y := A^T X + XA$ satisfies $Y < 0$ on $Ker(C)$. Let $\alpha > 0$ and $U = [U_1\ U_2] \in \mathbb{R}^{n\times n}$ be a unitary matrix such that the columns of U_1 span $Ker(C)$, and the relations $U_1^T Y U_1 < 0$, $U_2^T C^T C U_2 > 0$ hold. Then

$$U^T(\alpha Y - C^T C)U = \begin{pmatrix} \alpha U_1^T Y U_1 & \alpha U_1^T Y U_2 \\ \alpha U_2^T Y U_1 & \alpha U_2^T Y U_2 - U_2^T C^T C U_2 \end{pmatrix}.$$

Since $\alpha U_1^T Y U_1 < 0$, the matrix $U^T(\alpha Y - C^T C)U$ is negative definite if its Schur complement

$$-U_2^T C^T C U_2 + \alpha \left(U_2^T Y U_2 - U_2^T Y U_1 (U_1^T Y U_1)^{-1} U_1^T Y U_2 \right)$$

is negative definite, which is true for α sufficiently small. For this appropriate choice of α we then conclude that $P := \alpha X$ is a symmetric and positive definite solution to (4.1). □

As a complementary result to Lemma 5.4, we recall [15, Theorem 2.4.10].

Lemma 5.5: Consider the LQ-problem (2.1), (2.2). Assume that A does not have eigenvalues μ with $Re(\mu) = 0$. Then there exists a symmetric matrix P solution to (4.1), which is positive definite if $Re(\mu) < 0$ holds for all eigenvalues μ of A.

6 Eigenvalue conditions for strict (pre-)dissipativity

In this section we use the results developed so far in order to derive if-and-only-if conditions for strict (pre-)dissipativity based on Lemma 5.4 and 5.5.

Theorem 6.1: Consider the LQ-problem (2.1), (2.2) with $Q = C^T C$. Then the following holds:

(i) The problem is strictly dissipative if and only if A does not have unobservable eigenvalues μ with $Re(\mu) \geq 0$, i.e., if (A, C) is detectable.

(ii) The problem is strictly pre-dissipative if and only if A does not have unobservable eigenvalues μ with $Re(\mu) = 0$.

In both cases, the storage function can be chosen of the form $\lambda(x) = x^T P x + q^T x$, for suitable $P \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^n$.

Proof. Since all properties under consideration are invariant under coordinate changes, by Eq. (6.8) in [22] we may assume that A and C are of the form

$$A = \begin{pmatrix} A_1 & 0 \\ A_3 & A_2 \end{pmatrix}, \quad C = (C_1 \, 0),$$

with $A_1 \in \mathbb{R}^{r \times r}$, $A_2 \in \mathbb{R}^{(n-r)\times(n-r)}$, $A_3 \in \mathbb{R}^{(n-r)\times r}$, $C_1 \in \mathbb{R}^{l \times r}$, $r \in \{0, \ldots, n\}$ being the rank of the observability matrix $\mathbf{O}(A, C)$, and (A_1, C_1) being observable. Then $Q = C^T C$ is of the form

$$Q = \begin{pmatrix} Q_1 & 0 \\ 0 & 0 \end{pmatrix}$$

with $Q_1 = C_1^T C_1 \in \mathbb{R}^{r \times r}$. We may thus apply Lemma 5.4 in order to obtain a symmetric and positive definite matrix $P_1 \in \mathbb{R}^{r \times r}$ such that $Q_1 - A_1^T P_1 - P_1 A_1 > 0$.

(a) Now assume that one of the eigenvalue conditions in (i) or (ii) holds. Since all unobservable eigenvectors of A must be eigenvectors of A_2, we obtain that A_2 does not have eigenvalues μ with $Re(\mu) = 0$. Hence, we may apply Lemma 5.5 to $A = A_2$ and $Q = 0$ in order to obtain a symmetric matrix $P_2 \in \mathbb{R}^{(n-r)\times(n-r)}$ with $-A_2^T P_2 - P_2 A_2 > 0$. Here, P_2 is positive definite in case the eigenvalue condition from (i) holds.

For $\alpha > 0$, define $P_\alpha := \mathrm{diag}(P_1, \alpha P_2) \in \mathbb{R}^{n\times n}$. A straightforward computation yields

$$Q - A^T P_\alpha - P_\alpha A = \begin{pmatrix} Q_1 - A_1^T P_1 - P_1 A_1 & -\alpha A_3^T P_2 \\ -\alpha P_2 A_3 & -\alpha A_2^T P_2 - \alpha P_2 A_2 \end{pmatrix}.$$

Since $Q_1 - A_1^T P_1 - P_1 A_1 > 0$, we can conclude that $Q - A^T P_\alpha - P_\alpha A$ is positive definite if its Schur complement

$$-\alpha A_2^T P_2 - \alpha P_2 A_2 - \alpha^2 P_2 A_3 (Q_1 - A_1^T P_1 - P_1 A_1)^{-1} A_3^T P_2$$

is so. Since $-A_2^T P_2 - P_2 A_2$ is positive definite, the Schur complement is positive definite whenever $\alpha > 0$ is sufficiently small. Fixing such a sufficiently small $\tilde{\alpha} > 0$ and setting $P = P_{\tilde{\alpha}}$ we can apply Lemma 4.1 in order to conclude strict dissipativity if $P > 0$, i.e., in case (i), and strict pre-dissipativity in case (ii).

(b) Conversely, assume that the system is strictly dissipative at an equilibrium (x^e, u^e) and that the eigenvalue condition in (i) does not hold. Thus, let $\phi \in \mathbb{C}^n \setminus \{0\}$ be an unobservable eigenvector with eigenvalue μ satisfying $Re(\mu) \geq 0$. Let w denote the real part of ϕ, and set $x_0 = x^e + \gamma w$ for some $\gamma \in \mathbb{R}$ that will be specified below. Consider the solution $x_u(\cdot, x_0)$ corresponding to some control $u \in \mathbb{U}^\infty(x_0)$. Thanks to the linearity of the dynamics and relation (5.1), $x_u(\cdot, x_0)$ is decomposed as

$$x_u(t, x_0) = x_u(t, x^e) + x_0(t, \gamma w) = x_u(t, x^e) + \gamma e^{tA} w\,, \qquad \forall t \geq 0\,.$$

In particular, for $u = u^e$, we have that

$$x(t) := x_{u^e}(t, x_0) = x^e + \gamma e^{tA} w\,, \qquad \forall t \geq 0\,.$$

In case of μ real, we have that $\|x(t) - x^e\| = |\gamma|\, e^{\mu t}\, \|\phi\|$; in case of μ complex, we can appeal to the estimate from below in (A.2) in the Appendix, that yields the existence of a constant $m > 0$ such that $\|x(t) - x^e\| = |\gamma|\; \|e^{tA} w\| \geq |\gamma|\, e^{Re(\mu)t}\, m$. Thus in both cases we can choose $|\gamma|$ sufficiently large to ensure that there exists $\delta > 0$ such that $\alpha(\|x(t) - x^e\|) \geq \delta$ for all $t \geq 0$. On the other hand, the definition of unobservable eigenvectors implies the condition $Q e^{tA} w = 0$ for all $t \geq 0$, thus

$$\begin{aligned} \ell(x(t), u^e) &= x(t)^T Q x(t) + (u^e)^T R u^e + s^T x(t) + v^T u^e + c \\ &= (x^e)^T Q x^e + (u^e)^T R u^e + s^T x^e + s^T \gamma e^{tA} w + v^T u^e + c \\ &= \ell(x^e, u^e) + \gamma s^T e^{tA} w\,. \end{aligned} \tag{6.1}$$

We now choose the sign of γ such that $\gamma s^T w \leq 0$. Then in the real case a straightforward computation and in the complex case the application of Lemma A(ii) yields that there exist arbitrarily large $t > 0$ with

$$\int_0^t \gamma s^T e^{\tau A} w \,\mathrm{d}\tau \leq 0\,.$$

We can thus construct a sequence of $t_k \nearrow +\infty$ as $k \to +\infty$, such that the previous inequality holds for $t = t_k$. For these t_k, the strict dissipativity inequality together with identity (6.1) and with the relation $\alpha(\|x(t) - x^e\|) \geq \delta$ implies

$$\lambda(x(t_k)) \leq \lambda(x_0) + \int_0^{t_k} \gamma s^T e^{\tau A} w \, \mathrm{d}\tau - \delta t_k \leq \lambda(x_0) - \delta t_k \,.$$

Since this holds for t_k arbitrarily large, we deduce that $\lambda(x(t_k))$ tends to $-\infty$ for $k \to \infty$, which contradicts the boundedness of λ from below in the strict dissipativity assumption.

(c) Finally, assume that the eigenvalue condition in (ii) does not hold and assume the problem is strictly pre-dissipative. With the same construction as in point (b) we find a solution $x(t)$ starting in x_0, such that

$$\lambda(x(t_k)) \leq \lambda(x_0) - \delta t_k$$

for arbitrarily large $t_k \in \mathbb{R}$ and for some $\delta > 0$. If $x(t_k) = x_0$ holds for one of these t_k, this leads to the contradiction $\lambda(x_0) \leq \lambda(x_0) - \delta t_k$. In case of $x(t_k) \neq x_0$ for all k, we obtain that $\lambda(x(t_k))$ is unbounded from below for $k \to \infty$. In order to contradict the strict pre-dissipativity assumption, we have to show that $(x(t_k))_{k \in \mathbb{N}}$ belongs to a bounded set. Indeed, in case of μ real, since $Re(\mu) = 0$ we obtain $\|\gamma e^{tA} w\| = |\gamma| \, \|w\|$ for all $t \geq 0$, and thus $x(t_k)$ is contained in the closed ball centered at x^e with radius $|\gamma| \, \|w\|$. A similar argument holds in the case of μ complex, since from Lemma A(i) there exists $M > 0$ such that $\|\gamma e^{tA} w\| \leq M$ for every $t \geq 0$, thus $x(t_k)$ is contained in the closed ball centered at x^e with radius M, and $\lambda(x(t_k))$ is unbounded from below in this bounded set. This contradicts the requirement that λ is bounded from below on compact sets. □

7 Turnpike implies strict (pre-)dissipativity

We now have all the ingredients to state and prove the converse results to those from Section 3, which we already announced there.

Theorem 7.1: Consider the LQ-problem (2.1), (2.2) with $Q = C^T C$ and state and control constraint sets $\mathbb{X} \subseteq \mathbb{R}^n$ and $\mathbb{U} \subset \mathbb{R}^m$. Let $(x^e, u^e) \in \mathbb{X} \times \mathbb{U}$ be an equilibrium. Then the following holds:

(i) If $\mathbb{X} = \mathbb{R}^n$ and the problem has the turnpike property at (x^e, u^e), then it is strictly dissipative at (x^e, u^e).

(ii) If $\mathbb{X} \times \mathbb{U}$ contains a ball around (x^e, u^e) and the problem has the near equilibrium turnpike property at (x^e, u^e), then it is strictly pre-dissipative at (x^e, u^e).

Proof. (i) The proof proceeds by contraposition, i.e., we show that if strict dissipativity does not hold and $\mathbb{X} = \mathbb{R}^n$, then the turnpike property cannot hold. To this end, assume strict dissipativity does not hold. Then by Theorem 6.1 there exists an unobservable eigenvalue μ with $Re(\mu) \geq 0$. Let $w + iv$ be the corresponding eigenvector and set

$$x_0 = \frac{a}{(a^2 + b^2)^{1/2}} w + \frac{b}{(a^2 + b^2)^{1/2}} v \,.$$

Since for $\mathbb{X} = \mathbb{R}^n$ all solutions are feasible, from Remark 5.3 we know that the optimal control u^* for initial condition λx_0 is independent of $\lambda \in \mathbb{R}$. The explicit solution formula (5.1) implies that for $\lambda_1 \neq \lambda_2$ and all $t \geq 0$ and $u \in \mathbb{U}^T := \{(u(t))_{t\in[0,T]} : u(t) \in \mathbb{U}\ \forall t \in [0,T]\}$ the (in)equalities

$$\|x_u(t, \lambda_1 x_0) - x_u(t, \lambda_2 x_0)\| = |\lambda_1 - \lambda_2|\, \|e^{tA} x_0\| \geq |\lambda_1 - \lambda_2|\, C$$

hold, where $C = \|x_0\|$ if μ is real and $C = m > 0$ from Lemma A(i) otherwise. Since this in particular holds for the optimal controls, the turnpike property can hold for at most one of the two initial conditions. This contradicts Definition 2.1, which demands the property for all initial conditions in a bounded set.

(ii) Again, we show the implication by contraposition. Assume that strict pre-dissipativity does not hold. Then by Theorem 6.1 there exists an unobservable eigenvalue μ with $Re(\mu) = 0$. Let $w + iv$ be the corresponding eigenvector and set $x_0 = x^e + \lambda w$ for $\lambda \in \mathbb{R}$. Then for the control $u \equiv u^e$ we obtain

$$x_u(t, x_0) = x_u(t, x^e) + x_0(t, \lambda w) = x^e + \lambda e^{tA} w\,. \tag{7.1}$$

In case $b \neq 0$, from Lemma A(i) we obtain that

$$m \leq \|e^{tA} w\| \leq M \tag{7.2}$$

for all $t \geq 0$, with $M \geq m > 0$. In case $b = 0$, the same inequalities hold with $m = M = \|w\|$. Hence, since $\mathbb{X} \times \mathbb{U}$ contains a ball around (x^e, u^e), for $|\lambda|$ sufficiently small we have that $x^e + \lambda e^{tA} w$ lies in $\mathbb{X}$ for any $t \geq 0$. Moreover, the same calculation as that for (6.1) leads to

$$\ell(x_{u^e}(t, x_0), u^e) = \ell(x^e, u^e) + \lambda s^T e^{tA} w.$$

Thus, choosing $|\lambda|$ sufficiently small and with appropriate sign such that $\lambda s^T w \leq 0$, from Lemma A(ii) we obtain that

$$J_T(x_0, u^e) = T\ell(x^e, u^e) + \lambda \int_0^T s^T e^{tA} w \,\mathrm{d}t \leq T\ell(x^e, u^e)$$

for arbitrarily large T. However, because of (7.2) we obtain

$$\|x_u(t, x_0) - x^e\| = \|\lambda e^{tA} w\| \geq |\lambda| \min\{\|w\|, m\} > 0 \qquad \forall t \geq 0\,.$$

This implies that the near optimal turnpike property does not hold. □

8 The main equivalence results

In this section we summarize the results obtained so far by integrating them into two theorems, one for the case without state constraints and one for the case with state constraints. For both cases, we also provide several illustrative examples. We start by considering the case without state constraints.

Theorem 8.1: Consider the LQ-problem (2.1), (2.2) with (A, B) stabilizable, $Q = C^T C$ and state and control constraint sets $\mathbb{X} = \mathbb{R}^n$ and $\mathbb{U} \subseteq \mathbb{R}^m$. Then the following properties are equivalent

(i) The problem is strictly dissipative at an equilibrium $(x^e, u^e) \in \mathrm{int}\,(\mathbb{X} \times \mathbb{U})$.

(ii) The problem has the turnpike property at an equilibrium $(x^e, u^e) \in \mathrm{int}\,(\mathbb{X} \times \mathbb{U})$.

(iii) The pair (A, C) is detectable, i.e., all unobservable eigenvalues μ of A satisfy $Re(\mu) < 0$.

Moreover, if one of these properties holds, then the equilibria in (i) and (ii) coincide. If, in addition, $\mathbb{U} = \mathbb{R}^m$ holds, then the exponential turnpike property holds.

Proof. "(i) $\Rightarrow$ (ii)" follows from Corollary 3.2(i), "(ii) $\Rightarrow$ (i)" follows from Theorem 7.1(i), and "(i) $\Leftrightarrow$ (iii)" follows from Theorem 6.1(i). Moreover, the fact that the equilibria coincide follows from Corollary 3.2 and Theorem 7.1. Finally, the exponential turnpike property in the case of unconstrained inputs follows from [21, Corollary 3.2]. □

Figure 8.1 gives a schematic overview about the statements of Theorem 8.1 and the related results used in its proof.

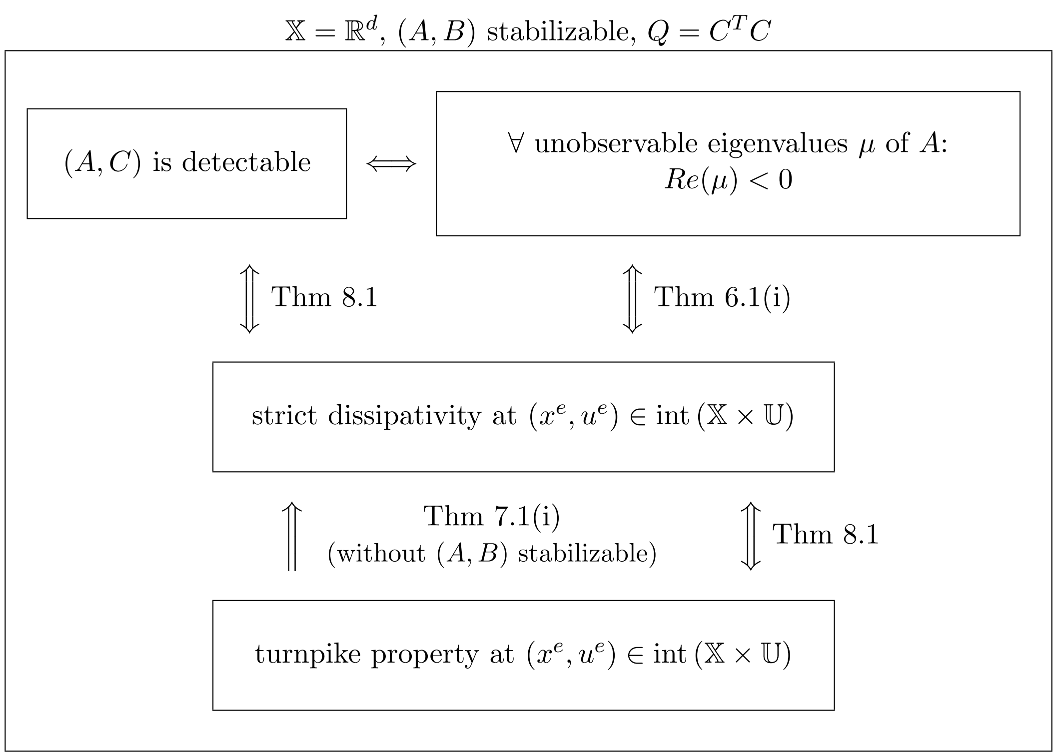

Figure 8.1: Schematic sketch of Theorem 8.1

We illustrate the application of Theorem 8.1 by some examples.

Example 8.2: Consider the LQ problem

$$\dot{x}(t) = x(t) + u(t)\,, \qquad t \in (0, T)$$

with $\mathbb{X} = \mathbb{U} = \mathbb{R}$, and stage cost $\ell(x, u) = x^2 + 0.005u^2$. Following the previous notations, we have that $A = B = I$, thus (A, B) is stabilizable (as a matter of fact, controllable).

Moreover, the eigenvalue $\mu = 1$ of A is observable, since $C = Q = I$, and thus the pair (A, C) is detectable. Thus, the exponential turnpike property at the equilibrium $(x^e, u^e) = (0, 0)$ holds.

Example 8.3: Consider the LQ problem

$$\dot{x}(t) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t), \qquad t \in (0, T)$$

with $\mathbb{X} = \mathbb{R}^2$, $\mathbb{U} = [-10, 10]$, and stage cost

$$\ell(x, u) = x_2^2 + 0.005u^2, \qquad x = (x_1, x_2).$$

With reference to the previous notations, $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, thus the pair (A, B) is stabilizable (but not controllable). Moreover, since $C = Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, the unobservable space, spanned by the eigenvector $x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, corresponds to the eigenvalue $\mu = -1$ of A, which has negative real part, and thus the pair (A, C) is detectable. Thus Theorem 8.1 ensures that the turnpike property holds for the system.

The second theorem summarizes our results for bounded state constraint set $\mathbb{X}$.

Theorem 8.4: Consider the LQ-problem (2.1), (2.2) with $Q = C^T C$ and state and control constraint sets $\mathbb{X} \subset \mathbb{R}^n$ bounded and $\mathbb{U} \subseteq \mathbb{R}^m$. Then the following properties are equivalent

(i) The problem is strictly pre-dissipative at an equilibrium $(x^e, u^e) \in \operatorname{int}(\mathbb{X} \times \mathbb{U})$.

(ii) The problem has the near equilibrium turnpike property at an equilibrium $(x^e, u^e) \in \operatorname{int}(\mathbb{X} \times \mathbb{U})$.

(iii) All unobservable eigenvalues μ of A satisfy $Re(\mu) \neq 0$.

Moreover, if one of these properties holds, then the equilibria in (i) and (ii) coincide. In addition, if (A, B) is stabilizable, then the turnpike property holds.

Proof. "(i) $\Rightarrow$ (ii)" follows from Theorem 3.1(ii), "(ii) $\Rightarrow$ (i)" follows from Theorem 7.1(ii), and "(i) $\Leftrightarrow$ (iii)" follows from Theorem 6.1(ii). The fact that the equilibria coincide follows from Theorem 3.1(ii) and Theorem 7.1(ii). In case of (A, B) stabilizable, the turnpike property follows from Remark 2.2(iii). □

The statements of Theorem 8.4 and the results used in its proof are schematically depicted in Figure 8.2.

Again, we illustrate the theorem by an example.

Example 8.5: Consider the LQ problem

$$\dot{x}(t) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u(t), \qquad t \in (0, T)$$

with $\mathbb{X} = [-1, 1]^2$, $\mathbb{U} = [-4, 4]$. In this case, we have $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, thus the pair (A, B) is controllable. We consider two different stage cost functions:

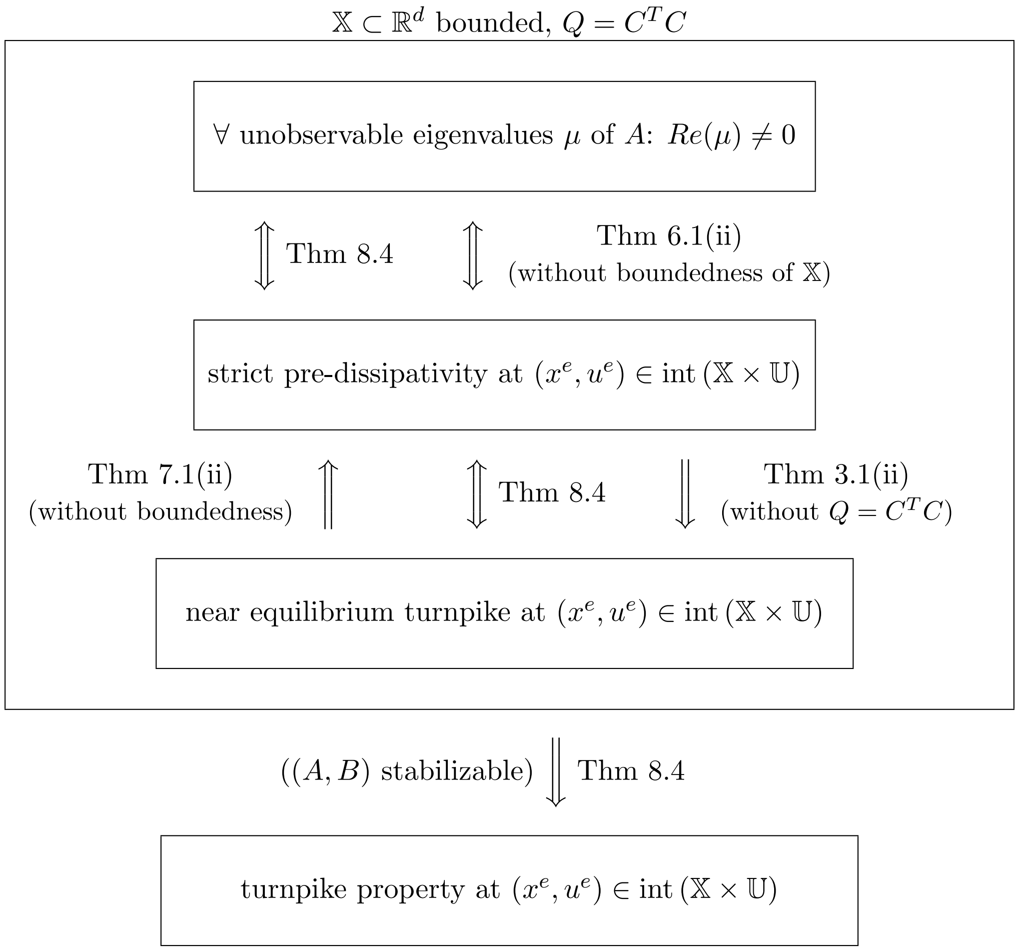

Figure 8.2: Schematic sketch of Theorem 8.4

i) $\ell_1(x, u) = \|x\|^2 + 0.005u^2\,, \qquad x = (x_1, x_2)\,.$
In this case, we have that $C = Q = I$, thus all eigenvalues of A are observable, and so Theorem 8.4 ensures that the turnpike property holds for the system.

ii) $\ell_2(x, u) = x_2^2 + 0.005u^2\,, \qquad x = (x_1, x_2)\,.$
In this case, $C = Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and the unobservable eigenvector $\bar{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ corresponds to the eigenvalue $\bar{\mu} = 2$ of A, which has real part different from zero. For this reason, Theorem 8.4 ensures then that the turnpike property holds for the system.

An important feature of the results in Theorems 8.1 and 8.4 is that they provide conditions which are also necessary and not merely sufficient. Hence, we can also detect situations in which the turnpike property does not hold. We illustrate this fact in the next examples.

Example 8.6: Consider the LQ problem of Example 8.5, but without bounded state constraints, i.e., with $\mathbb{X} = \mathbb{R}^2$, and running cost $\ell_2(x, u) = x_2^2 + 0.005u^2\,,\ x = (x_1, x_2)\,.$

In this case, since the unobservable eigenvector $\bar{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ corresponds to the eigenvalue $\bar{\mu} = 2$ of A, which has real part different from zero, Theorem 6.1 ensures that the problem is strict pre-dissipative. However, since $\mathbb{X}$ is not bounded, Theorem 8.4 does not apply. On the other hand, since condition (iii) of Theorem 8.1 is violated, we conclude that the turnpike property does not hold for the system.

Example 8.7: Consider the LQ problem

$$\dot{x}(t) = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t)\,, \qquad t \in (0,T)$$

where $x(t) = (x_1(t), x_2(t))$, with $\mathbb{X} = [-5,5]^2$, $\mathbb{U} = [-10,10]$, and stage cost $\ell(x,u) = u^2$. Since $A = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, the pair (A,B) is controllable. Moreover, since $C = Q = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, the two eigenvalues $\mu_{1/2} = \pm i$ are unobservable, with $Re(\mu_{1/2}) = 0$. Thus, condition (iii) of Theorem 8.4 fails, and then we deduce that the system does not fulfill the near equilibrium turnpike property.

Example 8.8: Consider the LQ problem

$$\dot{x}(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} u(t)\,, \qquad t \in (0,T)$$

with $x(t) = (x_1(t), x_2(t), x_3(t))$, $u(t) = (u_1(t), u_2(t))$, $\mathbb{X} = [-2,2]^3$, $\mathbb{U} = [-10,10]^2$. Since $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$, the pair (A,B) is controllable. The eigenvalues of A are given by $\mu_1 = 1$ and $\mu_{2/3} = \pm i$. We consider two different stage cost functions:

(i) Choosing $\ell_1(x,u) = x_1^2 + u^2$, since $C = Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, then only the eigenvector $e_1 = (1,0,0)$ corresponding to μ_1 is observable, while the eigenvectors corresponding to $\mu_{2/3}$ are unobservable. Since $Re(\mu_{2/3}) = 0$, condition (iii) of Theorem 8.4 fails, thus the system does not fulfill the near equilibrium turnpike property.

(ii) On the other hand, choosing a stage cost penalizing either x_2 or x_3, such as $\ell(x,u) = x_2^2 + u^2$, then $C = Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and thus both eigenvectors corresponding to $\mu_{2/3}$ are observable, while the unobservable eigenvector e_1 has eigenvalue μ_1 with real part different from zero. From Theorem 8.4 we can then conclude that the system satisfies the turnpike property.

For the sake of completeness, in the next example we show the occurrence of the third situation mentioned in the introduction, where the location of the turnpike equilibrium may change depending on the constraint sets.

Example 8.9: Consider the LQ problem

$$\dot{x}(t) = 2x(t) + u(t)\,, \qquad t \in (0,T)\,,$$

with cost function $\ell(x,u) = u^2$, and with state constraints $\mathbb{X} = [a,b]$ with $0 < a < b$, and without control constraints, i.e. with $\mathbb{U} = \mathbb{R}$. It is clear that the pair (A,B) is controllable. Moreover, the optimal equilibrium is $(x^e = a, u^e = -2a)$, because the cost for staying in any $x \in [a,b]$ is $u^2 = (-2x)^2 = 4x^2$, hence is minimal for $x = a$. Thus, the location of the optimal equilibrium depends on the choice of the state constraint set $\mathbb{X}$. As already mentioned in the Introduction, this situation is not covered by the results developed in this paper, and it is currently an open question whether it can be addressed by dissipativity techniques. This question will be investigated in future research.

A Appendix

This appendix provides a technical lemma which was needed in several proofs throughout this paper.

Lemma A: Let $A \in \mathbb{R}^{n\times n}$ and $\phi = w + iv$, $w, v \in \mathbb{R}^n$, be an eigenvector of A corresponding to the eigenvalue $\mu = a + ib \in \mathbb{C}$.

(i) For every c_0, $d_0 \in \mathbb{R}$,

$$e^{tA}(c_0 w + d_0 v) = e^{at}(c_t w + d_t v) \quad \forall t \geq 0\,, \tag{A.1}$$

with $c_t^2 + d_t^2 = c_0^2 + d_0^2$. Moreover, if $b \neq 0$, there are constants $M \geq m > 0$ such that for every c_0, $d_0 \in \mathbb{R}$ with $c_0^2 + d_0^2 = 1$ and for every $t \geq 0$

$$e^{at} m \leq \|e^{tA}(c_0 w + d_0 v)\| \leq e^{at} M\,. \tag{A.2}$$

(ii) Let $b \neq 0$ and $a \geq 0$, $x(t) := e^{tA} w$ for all $t \geq 0$ and $s \in \mathbb{R}^n$ satisfying $s^T w \leq 0$. Then there exist arbitrarily large $t > 0$ for which the inequality

$$\int_0^t s^T x(\tau)\,\mathrm{d}\tau \leq 0 \tag{A.3}$$

holds.

Proof. i) From the identity $A\phi = \mu\phi$ follows that

$$e^{tA}\phi = e^{t\mu}\phi\,, \qquad \forall t \geq 0\,.$$

Then a straightforward computation gives that, for all $t \geq 0$,

$$e^{tA} w = e^{at}\left(cos(bt)w - sin(bt)v\right)\,, \qquad e^{tA} v = e^{at}\left(sin(bt)w + cos(bt)v\right)\,, \tag{A.4}$$

thus (A.1) holds for all $t \geq 0$ with

$$c_t = c_0\cos(bt) + d_0\sin(bt)\,, \quad d_t = d_0\cos(bt) - c_0\sin(bt)\,,$$

that satisfy $c_t^2 + d_t^2 = c_0^2 + d_0^2$. In order to prove (A.2), observe that from (A.1) we obtain

$$\|e^{tA}(c_0 w + d_0 v)\|^2 = e^{2at}\|c_t w + d_t v\|^2\,.$$

It thus suffices to show the existence of $M \geq m > 0$ with $m^2 \leq \|cw + dv\|^2 \leq M^2$ for all $c, d \in \mathbb{R}$ with $c^2 + d^2 = 1$. For the squared Euclidean norm it holds that

$$\|cw + dv\|^2 = c^2\|w\|^2 + d^2\|v\|^2 + 2c\,d\langle w, v\rangle .$$

Since $2\langle w, v\rangle \leq \|w\|^2 + \|v\|^2$ and $|c| \leq 1$ and $|d| \leq 1$, we obtain the upper bound $M = 2(\|w\|^2 + \|v\|^2)$.

In order to find the lower bound $m > 0$, let $c_* \in \mathbb{R}$, $d_* \in \mathbb{R}$ with $c_*^2 + d_*^2 = 1$ be such that

$$\min_{c^2+d^2=1} c^2\|w\|^2 + d^2\|v\|^2 + 2cd\langle w, v\rangle = c_*^2\|w\|^2 + d_*^2\|v\|^2 + 2c_*d_*\langle w, v\rangle =: m.$$

Clearly, this m is a lower bound and it thus remains to show $m > 0$. To this end, If either $c_* = 0$ or $d_* = 0$ the assertion follows because $m = \|v\|$ or $m = \|w\|$, respectively. Otherwise, we set $w_* = c_* w$ and $v_* = d_* v$. Then, since $b \neq 0$, w and v span a two dimensional subspace (the sum of the eigenspaces corresponding to the complex conjugate eigenvalues μ and $\bar{\mu}$). Thus in the Cauchy-Schwarz inequality the strict inequality $|\langle w_*, v_*\rangle| < \|w_*\|\,\|v_*\|$ holds, because equality can only hold if w_* and v_* and thus w and v are linearly dependent. This yields

$$m = \|w_*\|^2 + \|v_*\|^2 + 2\langle w_*, v_*\rangle > \|w_*\|^2 + \|v_*\|^2 - 2\|w_*\|\,\|v_*\| = (\|w_*\| - \|v_*\|)^2 \geq 0$$

and thus the claim $m > 0$.

ii) Since s and A are real, from (A.4) we obtain

$$s^T x(\tau) = s^T e^{\tau A} w = s^T e^{a\tau}(\cos(b\tau)w - \sin(b\tau)v)\,,$$

which implies

$$\int_0^t s^T x(\tau)\,\mathrm{d}\tau = s^T \int_0^t e^{a\tau}\left[\cos(b\tau)w - \sin(b\tau)v\right]\mathrm{d}\tau\,. \tag{A.5}$$

We thus have to show that the right-hand side of this expression is non-positive for arbitrarily large t. If $a = 0$, (A.5) implies that

$$\int_0^t s^T x(\tau)\,\mathrm{d}\tau = \frac{\sin(bt)}{b} s^T w + \frac{\cos(bt) - 1}{b} s^T v\,,$$

thus the integral on the left-hand side is zero for any $t_k := \frac{2\pi}{b}k$, $k \in \mathbb{N}$, and (A.3) holds. If $a > 0$, because of the relations

$$\int e^{a\tau}\cos(b\tau)d\tau = \frac{e^{a\tau}}{a^2 + b^2}\left[b\sin(b\tau) + a\cos(b\tau)\right]\,,$$

$$\int e^{a\tau}\sin(b\tau)d\tau = \frac{e^{a\tau}}{a^2 + b^2}\left[a\sin(b\tau) - b\cos(b\tau)\right]\,,$$

from (A.5) we deduce

$$\int_0^t s^T x(\tau)\,\mathrm{d}\tau = \frac{1}{a^2 + b^2}\left[be^{at}\sin(bt) + ae^{at}\cos(bt) - a\right] s^T w + \frac{1}{a^2 + b^2}\left[be^{at}\cos(bt) - ae^{at}\sin(bt) - b\right] s^T v\,.$$

Since $a > 0$, for large values of t the dominant terms in the expressions

$$be^{at}\sin(bt) + ae^{at}\cos(bt) - a\,, \qquad be^{at}\cos(bt) - ae^{at}\sin(bt) - b$$

are those in e^{at}, with coefficients

$$c_w := b\sin(bt) + a\cos(bt)\,, \qquad c_v := b\cos(bt) - a\sin(bt)\,,$$

respectively. Now, observe that c_v is zero whenever $\sin(bt) = b/a\cos(bt)$, which holds true for any $t_k := 1/b\arctan(b/a) + \pi k/b$, for any $k \in \mathbb{Z}$. Moreover, for t_k of such form we have that $c_w = \frac{b^2+a^2}{a}\cos(bt_k)$. Then the dominant term in the coefficient of $s^T w$ is positive whenever $\cos(bt_k)$ is positive. Since $\arctan(b/a) \in (-\pi/2, \pi/2)$, then $\cos(bt_0) = \cos(\arctan(b/a)) > 0$. We thus conclude that $\cos(bt_k) > 0$ for the subsequence satisfying $b\bar{t}_k := \arctan(b/a) + 2\pi k$. For such sequence $(\bar{t}_k)_{k\in\mathbb{N}}$ the dominant term in the coefficient of $s^T w$ is positive, while $c_v = 0$. Then, since $s^T w < 0$ and $\bar{t}_k \nearrow +\infty$ as $k \to +\infty$, we have constructed a sequence of arbitrarily large times t satisfying (A.3). □

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